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Let X be any given base space and let $F(X)$ denote the class of all fuzzy subsets of X, $P(X)$ the class of all ordinary subsets of X, and $S(X)$ the class of all random subsets of X. It is now known that a partitioning of $S(X)$ exists whose components $S_A(X)$ are indexed by arbitrary $A \in F(X)$, with equivalence for each component determined by the one point coverage relation $\phi_A(x) = \Pr(x \in S(A))$, for all $x \in X$, $S(A)$ arbitrary in $S_A(X)$, with $S : F(X) \to S(X)$ denoting any corresponding choice-mapping.

Suppose $\star$ is an n-ary operation on $F(X) \times \cdots \times F(X)$ and S (as above is given. $\star$ is said to induce a weak n-homomorphism if there exists $* : P(X) \times \cdots \times P(X) \to P(X)$ such that for any $A_1, \ldots, A_n \in F(X)$ and thus $S(A_k) \in S_{A_k}(X)$, $k = 1, \ldots, n$, a joint distribution of $(S(A_1), \ldots, S(A_n))$ exists such that for all $x \in X$, $\phi_\star(A_1, \ldots, A_n)(x) \equiv \Pr(x \in S(\star(A_1, \ldots, A_n))) = \Pr(x \in *(S(A_1), \ldots, S(A_n)))$.

Characterizations are obtained for the class of n-ary fuzzy set operations $\star$ which induce weak n-homomorphisms with corresponding $*$ satisfying nested cases of measurability, continuity, and general composibility.

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CHARACTERIZATIONS OF N-ARY FUZZY SET OPERATIONS WHICH INDUCE HOMOMORPHIC RANDOM SET OPERATIONS

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ABSTRACT

Let X be any given base space and let $F(X)$ denote the class of all fuzzy subsets of X, $P(X)$ the class of all ordinary subsets of X, and $S(X)$ the class of all random subsets of X. It is now known that a partitioning of $S(X)$ exists whose components $S_A(X)$ are indexed by arbitrary $A \in F(X)$, with equivalence for each component determined by the one point coverage relation $\phi_A(x) = \Pr(x \in S(A))$, for all $x \in X$, S(A) arbitrary in $S_A(X)$, with $S: F(X) \to S(X)$ denoting any corresponding choice-mapping.
Suppose $\oplus$ is an n-ary operation on $F(X) \times \cdots \times F(X)$ and S (as above is given. $\oplus$ is said to induce a weak n-homomorphism iff there exists $\star: P(X) \times \cdots \times P(X) \to P(X)$ such that for any $A_1, \ldots, A_n \in F(X)$, and thus $S(A_k) \in S_{A_k}(X)$, $k=1, \ldots, n$, a joint distribution of $(S(A_1), \ldots, S(A_n))$ exists such that for all $x \in X$,
$\phi_{\oplus}(A_1, \ldots, A_n)(x) \equiv \Pr(x \in S \oplus (A_1, \ldots, A_n)) = \Pr(x \in \star(S(A_1), \ldots, S(A_n)))$.
Characterizations are obtained for the class of all n-ary fuzzy set operations $\oplus$ which induce weak n-homomorphisms with corresponding $\star$ satisfying nested cases of measurability, continuity, and general composibility.

KEYWORDS

Fuzzy sets, level sets, homomorphisms, fuzzy set operations, (ordinary) set operations, random sets.

1. INTRODUCTION

With the introduction of Zadeh's basic paper in 1965 [21], a new approach to the modeling of uncertainty began. The fledgling discipline of fuzzy set theory has now grown to encompass well over two thousand papers. (See the 1800 references in [7], as of 1979) Applications of fuzzy set techniques have been made to a wide variety of subjects, too numerous to be mentioned here. (See, e.g., Dubois and Prade's recent text [2].) As early as 1971, Zadeh himself recognized the role that level sets[1] played with respect to fuzzy sets [22]. In 1975 Negoita and Ralescu [14] (see also [17]) obtained a representation theorem for fuzzy sets and certain of their operations. (See also [16] for another level set approach.) Independent of this development, Goodman [4] in an unpublished paper in 1976 derived a simple homomorphic relationship between fuzzy and random sets and their operations – in this case the levels of the set involved were randomized uniformly (See also Nguyen's reformulation of this result [15]. A listing of papers contrasting the classical probability and fuzzy set approaches is given in the introductory section of [6] Additional comments on the controversy may be found in [1], [3], [13], [18] – [20].)

The two recent (independent) papers of Höhle [9] and Goodman [6] establish that, at least from a formal viewpoint, there exist systematic connections between fuzzy set theory and its operations, and probability theory and corresponding operations, via the concept of random sets. In the latter paper, one of the connections (S_U-type) makes use of the basic level set mapping, used earlier in [4], while another (T_0-type) is related to the product probability measure obtained from statistically independent 0–1 marginal random variables with parameters related to a given fuzzy set. Specifically, it was shown (summarized in Theorem 1 of this paper) that for any given fuzzy subset A of space X, there always exist random subsets S(A) of X (in general, non-unique) such that A and S(A) are equivalent under all one point coverages, i.e., $\phi_A(x) = \Pr(x \in S(A))$, for all constant $x \in X$. Furthermore, most of the common fuzzy set operations, such as union, intersection, complimentation, subsetting, cartesian product, and functional transform correspond homomorphically under the mapping $A \to S(A)$, for all fuzzy A, to ordinary operations for random sets (summarized in Theorem 2 here). Consequently, these fuzzy set operations can be reinterpreted directly in terms of corresponding ordinary set operations among random sets. One application of this connection is to deductive reasoning, where both fuzzy set and probabilistic techniques can be compatibly utilized [5]. (See also, [6], Theorems 5-7.)

The thrust of this paper is to expand earlier unary and binary homomorphic relations between fuzzy and random set operations, by obtaining systematic characterizations for three classes of n-ary fuzzy set operations, which under the mapping $A \to S(A)$ for any fuzzy subset A of X, yield homomorphic images, i.e., ordinary set operations applicable to the random sets S(A):

1. generalized composition: binary operations are presented in Theorem 3, unary operations, in Theorem 4, and n-ary operations in Theorem 6);

2. continuous: Theorem 5 demonstrates the general case, Corollary 1 exhibits characterizations for statistically independent image random sets, and Corollary 2 is concerned with a certain simplified subclass of continuous operations);

3. measurable (Theorem 7).

In the case of binary fuzzy set operations leading to homomorphic random set compositions (Theorem 3), the characterizing structure for the membership function is a simple sum of at most four terms, each term being also an elementary combination of the individual component membership functions and/or a functional form that satisfies a constraint equation (Eq. (2)) – see also Examples 1 and 2 for further clarification. In turn, this leads to 16 subtypes of binary operations, all explicitly given in Table 1. These include as special cases fuzzy union, intersection, bounded summation, product, etc. For unary operations, on the other hand, only identity, fuzzy complement, the null set reducing operation, and the universal set expanding operation, as given in Table 2, lead to homomorphic random set compositions. Another relatively simple characterization is obtained in Corollary 1, for n-ary fuzzy set operations leading to continuous homomorphic random set operations, when the n random set factors are chosen to be statistically independent, and when the mapping $A \to S(A)$ is either of two types: $S(A) = S_U(A)$ or $S(A) = T_A$ (see Theorem 1). Part (iii) of Theorem 6 also exhibits a straight forward characterization – a sum of products of factor membership functions and/or their complements – for n-ary fuzzy set operations leading to homomorphic random set operations which are in the form of compositions and for which the individual random set factors are chosen to be statistically independent. Theorem 5 and a specialization of it, Corollary 2, present very general – and somewhat more complicated – characterizations for fuzzy set operations leading to arbitrary continuous random set image operations. In this case, a typical n-ary fuzzy set operation is shown to be determined by a (pointwise-dependent in structure) finite signed sum of probabilities corresponding to each random set factor cover of a given fixed finite subset of X which is also disjoint from another such subset of X. Theorem 6, which treats the general measurability situation can be interpreted as being a uniform limiting case of reapplications of Theorem 5.

2. PRELIMINARY DEFINITIONS AND RESULTS

Let X denote a given base space, $P(X)$ its power class; $G(X)$, often identified with the latter, the class of all membership functions of elements of $P(X)$; $F(X)$ the class of all fuzzy subsets A – identified with the class of fuzzy set (or subset) membership functions $\phi_A : X \to [0,1]$; $S(X)$ the class of all random subsets – identified with the class of all probability measures on $P(X)$, with suitably chosen σ-algebras containing the sets $C_{\{x\}} = \{B \mid x \in B \in P(X)\}$ $x \in X$. (See [6]). Let S', $S'' \in S(X)$ and $A \in F(X)$. Define $A \approx S'$ (or $S' \approx A$) by, $\phi_A(x) = \Pr(x \in S')$, for all $x \in X$. Define $A \lessapprox S'$ by, $\phi_A(x) \leqslant \Pr(x \in S')$, for all $x \in X$. Note '$\stackrel{df}{=}$' means "is defined to be equal to".

To provide necessary background, two previously established theorems are summarized and reformulated.

Theorem 1 – (Goodman [6], Theorem 3)

(i) $S(X) = \bigcup_{A \in F(X)} S_A(X)$,

where $S_A(X) \stackrel{df}{=} \{S(A) \mid S(A) \in S(X) \ \& \ A \approx S(A)\}$; $\bigcup$ indicating disjoint union.

(ii) Each $S_A(X)$ is nonvacuous, and in fact, $S_U(A), T_A \in S_A(X)$, where $S_U(A) \stackrel{df}{=} \phi_A^{-1}[U, 1]$, U a uniform random variable over $[0,1]$ is determined by ϕ_{T_A} which corresponds to the product probability measure of the statistically independent 0-1 random variables $\phi_{T_A}(x)$ with $\Pr(\phi_{T_A}(x) = 1) = \phi_A(x)$; all $x \in X$.

(iii) If $A \in P(X)$, then $S_U(A) = T_A = A$ and $\{A\} = S_A(X)$ ∎

If $W \in S(X)$, then $A(W) \in F(X)$ is defined by $\phi_{A(W)}(x) \stackrel{df}{=} \Pr(x \in W)$, $x \in X$. Note, by definition, $A(W) \approx W$. Note also, W, $S_U(A(W))$, $T_{A(W)} \in S_{A(W)}$ are in general all distinct.

Fuzzy set operations $\oplus$, $\cdot$, $\otimes$, proj_1, (λ) are defined as.

$$\phi_{A \oplus B}(x) \equiv \phi_A(x) + \phi_B(x) - \phi_A(x) \cdot \phi_B(x);$$
$$\phi_{A \cdot B}(x) \equiv \phi_A(x) \cdot \phi_B(x);$$
$$\phi_{A \otimes B}(x,y) = \phi_A(x) \cdot \phi_B(y);$$
$$\phi_{\text{proj}_1(D)}(x) = \sup_{y \in D} \phi(x,y);$$
$$\phi_{A (\lambda) B}(x) = \lambda\, \phi_A(x) + (1-\lambda)\, \phi_B(x);$$

λ is identified with $\phi_\lambda(x) \equiv \lambda$; $0 \leqslant \lambda \leqslant 1$.

(The fuzzy set operations $\cup, \cap$, X, $\dashv$, f( ), etc., are the usual ones.)

Theorem 2 – (Modification of Goodman [6], Theorem 5)
For any $A, B \in F(X)$, $C \in F(Y)$, $f: X \to Y$, $D \in F(X \times Y)$:

$$S_U(A \cup B) = S_U(A) \cup S_U(B)$$
$$S_U(A \cap B) = S_U(A) \cap S_U(B)$$
$$S_U(A \times C) = S_U(A) \times S_U(C)$$
$$S_U(X \dashv A) = X \dashv S_{1-U}(A)$$
(excluding the boundary $\phi_A^{-1}(1-U)$)
$$S_U(f(A)) \subseteq f(S_U(A))$$
$$S_U(f^{-1}(C)) = f^{-1}(S_U(C))$$
$$S_U(\text{proj}_1(D)) \approx \text{proj}_1(D) \lessapprox \text{proj}_1(S_U(D))$$
$$T_{A \oplus B} \approx T_A \cup T_B$$
$$T_{A \cdot B} \approx T_A \cap T_B$$
$$T_{A \otimes C} \approx T_A \times T_C$$
$$T_{X \dashv A} \approx X \dashv T_A$$
$$T_{f(A)} \lessapprox f(T_A)$$
$$T_{f^{-1}(C)} \approx f^{-1}(T_C)$$
$$T_{\text{proj}_1(D)} \approx \text{proj}_1(D) \lessapprox \text{proj}_1(T_D)$$
$$T_{A (\lambda) B} \approx (\lambda \cap T_A) \cup ((1-\lambda) \cap T_B),$$

where T_A, T_B, T_C are to be chosen to be mutually statistically independent. ∎

Let $S : F(X) \to S(X)$ be such that for each $A \in F(X)$, $S(A) \in S_A(X)$ for some fixed choice of $S(A)$. Then S is called a choice function.

We will be concerned with the specification of joint distributions of $(S(A_1), \ldots, S(A_n))$ for all finite n and all $A_1, \ldots, A_n$

$\in F(X)$. Thus, if a set-valued stochastic process $(S(A))_{A \in F(X)} \overset{df}{=} S$, indexed by $F(X)$, over $P(X)$, can be constructed, all marginal distributions of the S(A)'s will be specified and consistently defined. This indeed will be the case for two important examples of S : $S(A) = S_{U_A}$, where U_A is uniformly distributed over [0,1], $A \in F(X)$ and $S(A) = T_A$, $A \in F(X)$. For any pair of such random sets $S(A_1)$, $S(A_2)$ (considered as a bivariate marginal random object from S), the following constraint and relations must hold: For any $A_1, A_2 \in F(X)$ and $x \in X$,

$$\delta_2(A_1, A_2; x) \overset{df}{=} \Pr(x \in S(A_1) \;\&\; x \in S(A_2)). \tag{1}$$

δ_2 always satisfies the fundamental joint probability constraint

$$L_2(A_1, A_2; x) \leq \delta_2(A_1, A_2; x) \leq U_2(A_1, A_2; x)\ , \tag{2}$$

where

$$\begin{aligned} L_2(A_1, A_2; x) &\overset{df}{=} \max(0, \Pr(x \in S(A_1)) \\ &\qquad + \Pr(x \in S(A_2)) - 1) \\ &= \max(0, \phi_{A_1}(x) + \phi_{A_2}(x) - 1)\ , \\ &\overset{df}{=} \phi_{(A \boxdot B)}(x) \end{aligned} \tag{3}$$

$$\begin{aligned} U_2(A_1, A_2; x) &\overset{df}{=} \min(\Pr(x \in S(A_1)), \Pr(x \in S(A_2))) \\ &= \min(\phi_{A_1}(x), \phi_{A_2}(x)) \\ &= \phi_{A \cap B}(x)\ , \end{aligned} \tag{4}$$

using Theorem 1.

It should be remarked that L_2 and U_2 depend only on the marginal probabilities. For the <u>statistical independence</u> case, $\delta_2(A_1, A_2; x) = \Pr(x \in S(A_1)) \cdot \Pr(x \in S(A_2)) = \phi_{A_1}(x) \cdot \phi_{A_2}(x)$ <u>always</u> satisfies Eq. (2).

Then using the basic properties of marginal and joint random variables and Eqs. (1)-(4) and Theorem 1, the four possible joint probabilities for the events $x \in$ or $\notin S(A_1)$ and $x \in$ or $\notin S(A_2)$ – equivalently, for the 0–1 random variables $\phi_{S(A_1)}(x)$ and $\phi_{S(A_1)}(x)$ – can be determined:

$$\begin{aligned} &\Pr(x \in S(A_1) \;\&\; x \in S(A_2)) \equiv \Pr(x \in S(A_1) \cap S(A_2)) \\ &\equiv \Pr(\phi_{S(A_1)}(x) = 1 \;\&\; \phi_{S(A_2)}(x) = 1) = \delta_2(A_1, A_2, x) \end{aligned} \tag{5}$$

$$\begin{aligned} &\Pr(x \in S(A_1) \;\&\; x \notin S(A_2)) \equiv \Pr(x \in S(A_1) - S(A_2)) \\ &\equiv \Pr(\phi_{S(A_1)}(x) = 1 \;\&\; \phi_{S(A_2)}(x) = 0) \\ &\quad = \phi_{A_1}(x) - \delta_2(A_1, A_2; x) \end{aligned} \tag{6}$$

$$\begin{aligned} &\Pr(x \notin S(A_1) \;\&\; x \in S(A_2)) \equiv \Pr(x \in S(A_2) - S(A_1)) \\ &\equiv \Pr(\phi_{S(A_1)}(x) = 0 \;\&\; \phi_{S(A_2)}(x) = 1) \\ &\quad = \phi_{A_2}(x) - \delta_2(A_1, A_2, x) \end{aligned} \tag{7}$$

$$\begin{aligned} &\Pr(x \notin S(A_1) \;\&\; x \notin S(A_2)) \equiv \Pr(x \in X - (S(A_1) \cup S(A_2))) \\ &\quad \equiv \Pr(\phi_{S(A_1)}(x) = 0 \;\&\; \phi_{S(A_2)}(x) = 0) \\ &\quad = 1 - \phi_{A_1}(x) - \phi_{A_2}(x) + \delta_2(A_1, A_2; x), \end{aligned} \tag{8}$$

The above results may be extended to the joint distribution of any $\phi_{S(A_1)}(x), \ldots, \phi_{S(A_n)}(x)$, <u>subject</u> <u>to</u> <u>consistency</u> <u>conditions</u> (for joint 0–1 events again, see [6], Lemmas 1,2 and Theorem 3) Here, it is sufficient to specify for $j = 1, \ldots, n$, $\delta_j(A_1, \ldots, A_j) \overset{df}{=} \Pr(\phi_{S(A_1)}(x) = 1 \;\&\; \ldots \;\&\; \phi_{S(A_j)}(x) = 1)$, in order to obtain the entire joint distribution of $x \in$ or $\notin S(A_j)$, $j = 1, \ldots, n$; in which case any such joint probability, analogous to Eqs. (5)-(8) can be written as a simple positive and/or negative combination of δ_j's.

<u>Example</u> 1. The case $S = T_{(\cdot)}$.

By choosing, for any $A_1, A_2 \in F(X)$ and $x \in X$, $\delta_2(A_1, A_2, x)$ arbitrary fixed but satisfying the constraint in Eq. (2), this specifies the total joint distribution of the 0–1 random variables $(\phi_{T_{A_1}}(x), \phi_{T_{A_2}}(x))$. Keep x fixed. Then by a similar argument, $\Pr(\phi_{T_{A_1}}(x) = 1 \;\&\; \phi_{T_{A_2}}(x) = 1 \;\&\; \phi_{T_{A_3}}(x) = 1)$ may be chosen arbitrary subject to a constraint (see [6], Lemmas 1, 2 and Theorem 3 for a similar result), and the construction may be extended indefinitely, yielding all <u>consistent</u> (or projective) finite combinations of marginal probabilities for all $A_1, A_2, A_3 \in F(X)$. Then (e.g., see [12], pp. 92–95) a probability measure μ_x can be obtained for the base space $\underset{A \in F(X)}{\times} \{0, 1\} = \{0, 1\}^{F(X)} = G(F(X))$ (with a corresponding σ-algebra of subsets) such that for any $A_1, A_2 \in F(X)$, $(\phi_{T_{A_1}}(x), \phi_{T_{A_2}}(x))$ corresponds to the $(A_1, A_2)^{th}$ marginal distribution with respect to μ_x, etc.

Again note the special case $\delta_2(A_1, A_2, x) = \phi_{A_1}(x) \cdot \phi_{A_2}(x)$, for all $A_1, A_2 \in F(X)$, leads to a consistent construction of μ_x by choosing all marginal 0-1 random variables $\phi_{T_A}(x)$ to be statistically independent. In addition, note that for <u>any</u> construction of the T_A's, <u>we</u> <u>can</u> <u>always</u> <u>consider</u> <u>probability</u> <u>measure</u> <u>μ_x</u> <u>to</u> <u>correspond</u> <u>to</u> <u>mutually</u> <u>statistically</u> <u>independent</u> <u>probability</u> <u>measures</u>, for <u>each</u> <u>distinct</u> <u>$x \in X$</u>. Hence, in turn, product probability measure μ can be constructed from the μ_x's. Thus μ corresponds to a well-defined joint probabilistic specification of the process $(\phi_{T_A} \mid A \in F(X))$, or equivalently, for $(T_A \mid A \in F(X))$, which is compatible with all original one- and two-level marginal distributions of the T_A's. Thus any given collection of fuzzy subsets can be made to correspond to a single joint probability measure which is compatible with the relations $A \approx T_A$ and possible additional bivariate distributional constraints on (T_A, T_B), for any $A, B \in F(X)$. ∎

Example 2. The case $S = S_U$.

As a second example, consider the situation when $S = S_U$. More precisely, consider $S(A) = S_{U_A}(A)$, for any $A \in F(X)$, where U_A is a uniformly distributed random variable over [0, 1]. Let Ψ be the probability distribution function for the standardized normal distribution n (0, 1). Define the transform τ by:

$$\tau : \underset{A \in F(X)}{\times} \mathbb{R} \cong \mathbb{R}^{F(X)} \to \underset{A \in F(X)}{\times} [0,1] \cong [0,1]^{F(X)} \cong FF(X)\ ,$$

where for any

$$f = (x_A)_{A \in F(X)} \in \mathbb{R}^{F(X)},\ \tau(f) \overset{\mathrm{df}}{=} (\Psi(x_A))_{A \in F(X)}.$$

Let $\gamma : F(X) \times F(X) \to M$, be any given marginally consistent correlation function where $M \overset{\mathrm{df}}{=} \left\{ \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \middle| -1 \leq r \leq 1 \right\}$ and let $m : F(X) \to \{0\}$ be the zero mean function. Then a unique gaussian stochastic process $V = (V_A)_{A \in F(X)}$ can always be constructed over $\mathbb{R}^{F(X)}$ corresponding to probability measure μ, etc., which has the specified γ and m. (See, e g. [10], for more details.) Then, since it readily follows that τ is a measurable mapping, $\tau(V) = (\Psi(V_A))_{A \in F(X)}$ is a stochastic process over $[0, 1]^{F(X)}$ with the following properties:

(i) For any $A \in F(X)$, the marginal random variable $\nu_A \overset{\mathrm{df}}{=} \Psi(V_A)$ is uniformly distributed over $[0, 1]$.

(ii) Subject to consistency for γ, for $n \geq 1$ and any $A_1, \ldots, A_n \in F(X)$, $x \in X$, $\delta_n(A_1, \ldots, A_n; x)$ can be made arbitrary within natural constraints (again see [6], Lemmas 1, 2 and Theorem 3). In particular, for $n = 2$, $\delta_2(A_1, A_2; x)$ may be chosen arbitrarily to satisfy Eq. (2) (for the usual consistency checks). (This follows directly from a continuity argument applied to the two boundary values in Eq. (2), which correspond to $r = 1$ (yielding $U_{A_1} = U_{A_2}$) and $r = -1$ (yielding $U_{A_1} = 1 - U_{A_2}$), for $\gamma(A_1, A_2)$.)

(iii) The following two special cases of (ii) are of particular importance:

(a) Choose all U_A, $A \in F(X)$ to be statistically independent and identically distributed – with common distribution being a uniform one over $[0, 1]$. Thus all $S_{U_A}(A)$, $A \in F(X)$, are statistically independent. (This corresponds to correlations $\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = I_2$.)

(b) Choose all $U_A \equiv U$, $A \in F(X)$. Thus all $S_{U_A}(A) \equiv S_U(A)$, $A \in F(X)$ are highly statistically dependent. (This corresponds to correlation matrices $\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.)

Finally, note that no matter what choice function S is used, for $n \geq 1$ fixed, $S(A_1), \ldots, S(A_n)$ can always be chosen to be statistically independent marginal random sets of the product probability measure $\mu = \underset{A \in F(X)}{\times} \mu_A$, each μ_A corresponding to $S(A)$.

3. THE BASIC PROBLEM

With the preliminary results established, we may now pose the basic problem. Let S be a general choice function and assume $(S(A))_{A \in F(X)}$ is some well-defined stochastic process over $P(X)$. Hence, $(S(A_1), \ldots, S(A_n))$ for all $A_1, \ldots, A_n \in F(X)$ can be assigned consistent joint distributions. Then for certain classes of ordinary set operations $\star : P(X) \times \cdots \times P(X) \to P(X)$ we seek corresponding classes of fuzzy set operations $\circledast$. $F(X) \times \cdots \times F(X) \to F(X)$ such that $\circledast$ induces a weak-n-homomorphism relative to $\star$, i.e.,

$$S(\circledast(A_1, \ldots A_n)) \approx \circledast(A_1, \ldots A_n)$$

$$\approx \star(S(A_1), \ldots, S(A_n)), \tag{9}$$

for all $A_1, \ldots, A_n \in F(X)$.

Conversely, given any fuzzy set operation $\circledast$, it is of interest to determine if some ordinary set operation $\star$ exists satisfying (9).

We will consider three general (nested) cases for $\star$: composition, continuous, measurable.

Consider first the composition case. We say ordinary set operation $\star$ is a generalized composition iff $h_k : X \to X$, $k = 1, \ldots, n$ and for each $x \in X$, $\exists\ g_x : \{0,1\} \times \cdots \times \{0,1\} \to \{0,1\}$ such that for all $B_1, \ldots, B_n \subset P(X)$,

$$\phi_{\star(B_1, \ldots, B_n)}(x) = g_x(\phi_{B_1}(h_1(x)), \ldots \phi_{B_n}(h_n(x))). \tag{10}$$

Note that for any given $B_1, \ldots, B_n, h_1, \ldots, h_n$, x as above $\star(B_1, \ldots, B_n)$ can be any of 2^{2^n} possible combinations for g_x, involving ordinary unions, intersections and complements of the sets $h_1^{-1}(B_1), \ldots, h_n^{-1}(B_n)$.

4. BINARY AND UNARY COMPOSITIONS

Theorem 3

Let $n = 2$ and S be a given choice function with all bivariate $(S(A_1), S(A_2))$ assumed to have specified consistent joint distributions.

(i) Then binary fuzzy set operation $\circledast$ induces a weak 2-homomorphism relative to some $\star$ being a generalized composition (as in Eq. (10)) iff $\circledast$ has the following form:

$$\begin{aligned} \phi_{\circledast(A_1, A_2)}(x) = {} & g_x(0,0) \cdot (1 - \phi_{A_1}(h_1(x)) - \phi_{A_2}(h_2(x)) \\ & + \delta_2(A_1, A_2; x)) \\ & + g_x(1,0) \cdot (\phi_{A_1}(h_1(x)) - \delta_2(A_1, A_2; x)) \\ & + g_x(0,1) \cdot (\phi_{A_2}(h_2(x)) - \delta_2(A_1, A_2; x)) \\ & + g_x(1,1) \cdot \delta_2(A_1, A_2; x), \end{aligned} \tag{11}$$

for all $A_1, A_2 \in F(X)$ and $x \in X$; $g_x(i,j) \in \{0,1\}$

(ii) By varying the distributions of $S(A_1), S(A_2))$ and the function $g_x : \{0,1\} \times \{0,1\} \to \{0,1\}$, the $\delta_2(A_1, A_2; x)$ can be chosen arbitrary fixed to satisfy Eq. (2), up to consistency. Note, in addition, that the statistical independence case $\delta_2(A_1, A_2, x) = \phi_{A_1}(x) \cdot \phi_{A_2}(x)$ always satisfies Eq. 2. In general, the coefficients $g_x(ij)$ may be arbitrary $\in \{0,1\}$, for $i,j \in \{0,1\}$: for all $A_1, A_2 \in F(X)$, $x \in X$.

(Proof
$$\begin{aligned} \phi_{\circledast(A_1, A_2)}(x) &= \Pr(x \in \star(S(A_1), S(A_2))) \\ &= \Pr(\phi_{\star(S(A_1), S(A_2))}(x) = 1) \\ &= \Pr(g_x(\phi_{S(A_1)}(h_1(x)), \phi_{S(A_2)}(h_2(x))) = 1) \\ &= \sum_{(i,j) \in g_x^{-1}(1)} \Pr(\phi_{S(A_1)}(h_1(x)) = i \\ &\qquad \& \ \phi_{S(A_2)}(h_2(x)) = j) \end{aligned}$$

which yields the form in Eq. (11), by again, use of Eqs (5)-(8).)

Note the application of Theorem 3 (via Examples 1 and 2 previously discussed) to the special cases: $S(A) = S_{U_A}(A)$, for either $U_A \equiv U$, or U_A mutually statistically independent, for all $A \in F(X)$, where U and each U_A are identically distributed uniform over $[0,1]$; and T_A.

Table 1 exhibits, for a given $S; x \in X; A_1, A_2 \in F(X): h_1, h_2$: the 16 possible types of binary fuzzy set operations $\circledast$ inducing weak homomorphic ordinary set operations $\star$ in compositional form. By following the comments at the end of Example 1, a similar table (now using δ_3 in addition to δ_2) of $2^{2^3} = 256$ entries may be constructed, etc. (See Theorem 6 for the general composition case.)

Thus we see that <u>several different</u> binary fuzzy set operations can induce the same homomorphic image – binary random set operation – for the same <u>fixed</u> choice function, by varying the statistical dependence (or stochastic process) between the random sets. ■

The analogue to Theorem 3 for unary operations is an important special case:

<u>Theorem 4</u>

Let n = 1 and S be a given choice function. Then unary fuzzy set operation $\circledast$ induces a weak 1-homomorphism relative to some $\star$ being a generalized composition <u>iff</u> $\circledast$ has the following form:

$$\phi_{\star(A)}(x) = g_x(0) \cdot (1 - \phi_A(h(x))) + g_x(1) \cdot \phi_A(h(x)), \tag{12}$$

where as $g_x : \{0,1\} \to \{0,1\}$ varies, $g_x(0), g_x(1)$ may be arbitrary $\in \{0,1\}$; for all $A \in F(X), x \in X$. In this situation, for any $x \in X, B \in P(X)$,

$$x \in \star(B) \text{ iff } x \in g(B)(x), \tag{13}$$

where for any $y \in X$,

$$g(B)(y) \overset{\text{df}}{=} \bigcup_{i \in g_y^{-1}(1)} \left(\underset{(\text{for } i=1)}{\cap h^{-1}(B)} \dashv \underset{(\text{for } i=0)}{\cup h^{-1}(B)} \right), \tag{14}$$

again using the convention $\bigcap_{\emptyset}(\,) = X$, $\bigcup_{\emptyset}(\,) = \emptyset$. ■

Table 1. Tabulation of Possible Binary Fuzzy Set Operations Inducing Weak Homomorphic Ordinary Set Operations in Compositional Form.

Case	$g_x^{-1}(1)$	$\star(S(A_1), S(A_2))$	$\phi_{\circledast}(A_1, A_2)(x)$
1	$\emptyset$	$\emptyset$	0
2	$\{(0,0)\}$	$X \dashv (h_1^{-1}(S(A_1)) \cup h_2^{-1}(S(A_2))$	$1 - \phi_{A_1}(h_1(x)) - \phi_{A_2}(x)) + \delta_2$
3	$\{(0,1)\}$	$h_2^{-1}(S(A_2)) \dashv h_1^{-1}(S(A_1))$	$\phi_{A_2}(h_2(x)) - \delta_2$
4	$\{(1,0)\}$	$h_1^{-1}(S(A_1)) \dashv h_2^{-1}(S(A_2))$	$\phi_{A_1}(h_1(x)) - \delta_2$
5	$\{(1,1)\}$	$h_1^{-1}(S(A_1)) \cap h_2^{-1}(S(A_2))$	δ_2
6	$\{(0,0),(0,1)\}$	$X \dashv h_1^{-1}(S(A_1))$	$1 - \phi_{A_1}(h_1(x))$
7	$\{(0,0),(1,0)\}$	$X \dashv h_2^{-1}(S(A_2))$	$1 - \phi_{A_2}(h_2(x))$
8	$\{(0,0),(1,1)\}$	$X \dashv (h_1^{-1}(S(A_1)) \Delta h_2^{-1}(S(A_2)))$	$1 - \phi_{A_1}(h_1(x)) - \phi_{A_2}(h_2(x)) + 2\delta_2$
9	$\{(0,1),(1,0)\}$	$h_1^{-1}(S(A_1)) \Delta h_2^{-1}(S(A_2))$	$\phi_{A_1}(h_1(x)) + \phi_{A_2}(h_2(x)) - 2\delta_2$
10	$\{(0,1),(1,1)\}$	$h_2^{-1}(S(A_2))$	$\phi_{A_2}(h_2(x))$
11	$\{(1,0),(1,1)\}$	$h_1^{-1}(S(A_1))$	$\phi_{A_1}(h_1(x))$
12	$\{(0,0),(0,1),(1,0)\}$	$X \dashv (h_1^{-1}(S(A_1)) \cap h_2^{-1}(S(A_2)))$	$1 - \delta_2$
13	$\{(0,0),(0,1),(1,1)\}$	$h_2^{-1}(S(A_2)) \cup (X \dashv h_1^{-1}(S(A_1)))$	$1 - \phi_{A_1}(h_1(x)) + \delta_2$
14	$\{(0,0),(1,0),(1,1)\}$	$h_1^{-1}(S(A_1)) \cup (X \dashv h_2^{-1}(S(A_2)))$	$1 - \phi_{A_2}(h_2(x)) + \delta_2$
15	$\{(0,1),(1,0),(1,1)\}$	$h_1^{-1}(S(A_1)) \cup h_2^{-1}(S(A_2))$	$\phi_{A_1}(h_1(x)) + \phi_{A_2}(h_2(x)) - \delta_2$
16	$\{(0,0),(0,1),(1,0),(1,1)\}$	X	1

$\delta_2 = \delta_2(A_1, A_2, x)$ may be arbitrary satisfying Eq. (2). g_x is assumed to be the same for all $x \in X$.

Table 2 lists the four possible unary fuzzy set operations leading to weak homomorphic ordinary set operations in composition form.

Table 2. Tabulation of Possible Unary Fuzzy Set Operations Inducing Weak Homomorphic Ordinary Operations in Compositional Form.

Case	$g_x^{-1}(1)$	$\star(S(A))$	$\phi_{\circledast}(A)(x)$
1	$\emptyset$	$\emptyset$	0
2	$\{0\}$	$X - h^{-1}(S(A))$	$1-\phi_A(h(x))$
3	$\{1\}$	$h^{-1}(S(A))$	$\phi_A(h(x))$
4	$\{0,1\}$	X	1

5. GENERAL RESULTS

The following more general results require some topological background. (See, e.g., [11] for all relevant definitions and general relations.)
Let $\mathscr{D}$ be the discrete topology on $\{0,1\}$. Thus the open sets for $\mathscr{D}$ consist of $P(\{0,1\})$. Then the natural topology $\mathscr{T}$ for $G(X)$, noting $G(X) = \underset{x\in X}{\times}\{0,1\} = \{0,1\}^X$, is the cartesian product (or, equivalently, pointwise convergence) topology. Open sets for $\mathscr{T}$ consist typically of all unions of sets of the form

$$\mathscr{O}(z,a) \overset{df}{=} \{\phi_B \mid \phi_B \in G(X) \;\&\; \phi_B(z_i) = a_i,\ \text{for } i = 1,\dots,q\}, \tag{15}$$

where

$$z \overset{df}{=} \begin{pmatrix} z_1 \\ \vdots \\ z_q \end{pmatrix},\quad a \overset{df}{=} \begin{pmatrix} a_1 \\ \vdots \\ a_q \end{pmatrix};\ z_i \in X,\ a_i \in \{0,1\};\ i = 1,\dots,q.$$

$z \in X \times\cdots\times X$, $a \in \{0,1\} \times\cdots\times \{0,1\}$ (q factors); $q \geq 1$, can all be chosen arbitrary. In turn, let $\mathscr{V}$ be the natural cartesian product topology for $G(X) \times\cdots\times G(X)$ (n factors). In this case, the open sets for $\mathscr{V}$ consist of all unions of sets of the form

$\mathscr{O}_1 \times\cdots\times \mathscr{O}_n$, where each $\mathscr{O}_i$ is arbitrary open for $\mathscr{T}$.

Note that for any ordinary set operation $\star$, there is a unique corresponding $\underset{\smile}{\star}$, $\underset{\smile}{\star} : G(X) \times\cdots\times G(X) \to G(X)$ (n factors). Then for any $x \in X$, $B_1,\dots,B_n \in P(X)$,

$$\phi_{\star(B_1,\dots,B_n)} = \underset{\smile}{\star}(\phi_{B_1},\dots,\phi_{B_n}), \tag{16}$$

and

$$\begin{aligned} x \in \star(B_1,\dots,B_n) \text{ iff } & \phi_{\star(B_1,\dots,B_n)}(x) = 1 \\ \text{iff } & (\underset{\smile}{\star}(\phi_{B_1},\dots,\phi_{B_n}))(x) = 1 \\ \text{iff } & (\phi_{B_1},\dots,\phi_{B_n}) \in \underset{\smile}{\star}^{-1}(\mathscr{O}(x,1)). \end{aligned} \tag{17}$$

In particular we can let $B_j = S(A_j)$; $A_j \in F(X)$; $j=1,\dots,n$. We say fuzzy set operation $\circledast$ induces a weak n-homomorphism relative to ordinary set operation $\star$ being continuous iff $\underset{\smile}{\star}$ is continuous with respect to topologies $(\mathscr{V},\mathscr{T})$ and Eq. (9) holds.
$\mathscr{D}$ is compact, Hausdorff, and second countable and separable. Consequently, the product topologies (by Tychnoff's Theorem) $\mathscr{T}$ and $\mathscr{V}$ are also compact and Hausdorff. If X an at most countably infinite space, then $\mathscr{T}$ and $\mathscr{V}$ will also be second countable and separable.

<u>Theorem</u> 5

Let $n \geq 1$ be arbitrary fixed and let S be a given choice function with all $(S(A_1),\dots S(A_n))$ assumed to have a specified <u>consistent</u> distribution.

(i) Then n-ary fuzzy set operation $\circledast$ induces a weak n-homomorphism relative to some ordinary set operation $\star$ being continuous <u>iff</u> $\circledast$ has the following form:

$$\phi_{\circledast}(A_1,\dots,A_n)(x) = \Pr\left(\underset{\nu\in K_x}{\text{Or}}\ \underset{j=1}{\overset{n}{\&}}\ (M_{\nu j} \subseteq S(A_j) \subseteq X - N_{\nu,j})\right)$$

$$\equiv \sum_{\substack{\text{over all } B,\\ \emptyset \neq B \in P(K_x),\text{ for}\\ M(B,j)\,\cap N(B,j) = \emptyset\\ \text{for all } j,\ 1 \leq j \leq n}} \left[(-1)^{\text{card}(B)+1} \cdot \Pr\left(\underset{j=1}{\overset{n}{\&}}\ (M(B,j) \subseteq S(A_j) \subseteq (X - N(B,j)))\right)\right]. \tag{18}$$

for all $A_1,\dots,A_n \in F(X)$ and $x \in X$, where not depending on the A_i's, $\exists$ finite (or vacuous) sets $M_{\nu j}, N_{\nu j} \in P(X)$ with $M(B,j) \overset{df}{=} \underset{\nu\in B}{\cup} M_{\nu,j}$, $N(B,j) \overset{df}{=} \underset{\nu\in B}{\cup} N_{\nu,j}$; for each $x \in X$, K_x is some finite index set.

(ii) Not only may the $(S(A_1),\dots, S(A_n))$ be varied in joint distribution, but also the $M_{\nu j}$'s and $N_{\nu j}$'s may be chosen arbitrary finite (or vacuous) $\in P(X)$; K_x may be chosen also as an arbitrary finite index set, for each $x \in X$. ∎

(<u>Proof</u>:

Using Eq. (17), for any ordinary continuous n-set operation $\star$, for any $B_1,\dots,B_n \in P(X)$ and $x \in X$,

$$x \in \star(B_1,\dots,B_n)$$

$$\text{iff } (\phi_{B_1},\dots,\phi_{B_n}) \in \underset{\nu\in K_x}{\cup}\ \underset{j=1}{\overset{n}{\times}}\ \mathscr{O}(Z_{\nu j}, a_{\nu j})$$

for some finite index set K_x, etc.

$$\text{iff } \underset{\nu\in K_x}{\text{Or}}\left(\underset{j=1}{\overset{n}{\&}}\ (M_{\nu,j} \subseteq B_j \subseteq X - N_{\nu j})\right), \tag{19}$$

where

$$M_{\nu j} \overset{df}{=} \{Z_{\nu j t} \mid 1 \leq t \leq q_{\nu j} \text{ such that } a_{\nu j t} = 1\} \tag{20a}$$

$$N_{\nu j} \overset{df}{=} \{Z_{\nu j t} \mid 1 \leq t \leq q_{\nu j} \text{ such that } a_{\nu j t} = 0\}.) \tag{20b}$$

∎

<u>Corollary</u> 1

Suppose for given choice function S, that, for any $A_1, \dots, A_n \in F(X)$, $S(A_1), \dots, S(A_n)$ are always mutually statistically independent. Then the characterization for $\circledast$ in Theorem 5 can be made to depend on $A_1,\dots,A_n$ through $\phi_{A_1}, \dots, \phi_{A_n}$ directly and not on probability statements involving $S(A_j)$, $j=1,\dots,n$, for any $A_1,\dots,A_n \in F(X)$, when either case 1 or case 2 holds for choice of S:

Case 1: For any $A \in F(X)$. $S(A) \equiv S_{U_A}(A)$, where as marginal random variables, each U_A is uniformly distributed over [0,1], but jointly all U_A's for different A's are mutually statistically independent. In this case, the characterizing equation, Eq. (18), becomes

$$\phi_{\circledast}(A_1,\dots,A_n)(x) = \sum_{\substack{\text{over all } B,\\ \emptyset \neq B \in P(K_x),\\ \text{for } M(B,j) \cap N(B,j) = \emptyset,\ \text{for all } j,\ 1 \leq j \leq n}} ((-1)^{\text{card } B+1} \cdot \prod_{j=1}^{n} \omega'(B, A_j)) , \quad (21)$$

$$\omega'(B, A_j) \stackrel{\text{df}}{=} \inf_{y \in M(B,j)} (\phi_{A_j}(y)) - \inf \{y \mid y \in X \,\&\, \phi_{A_j}^{-1}[y,1] \cap N(B,j) = \emptyset\} . \quad (22)$$

Case 2: For any $A \in F(X)$, $S(A) \equiv T_A$, where jointly all of the T_A's are mutually statistically independent. In this case, the characterizing equation, Eq. (18), becomes

$$\phi_{\circledast}(A_1,\dots,A_n)(x) = \sum_{\substack{\text{over all } B,\\ \emptyset \neq B \in P(K_x),\\ \text{for } M(B,j) \cap N(B,j) = \emptyset,\ \text{for all } j,\ 1 \leq j \leq n}} ((-1)^{\text{card } B+1} \cdot \prod_{j=1}^{n} \omega''(B, A_j)) , \quad (23)$$

$$\omega''(B, A_j) \stackrel{\text{df}}{=} \prod_{(y \in M(B,j))} \phi_{A_j}(y) \cdot \prod_{(y \in N(B,j))} (1 - \phi_{A_j}(y)) \cdot \quad (24)$$

■

Corollary 2

Let $n \geq 1$ be arbitrary and S a given choice function with all $(S(A_1), \dots, S(A_n))$ assumed to have a specified distribution.

(i) Then n-ary fuzzy set operation $\circledast$ induces a weak n-homomorphism relative to some ordinary set operation $\star$ being continuous such that the sum in Eq. (18) becomes a non-negative one, i.e., all terms for non-singleton B are vacuous, i.e.

$$\phi_{\circledast}(A_1, \dots A_n)(x)$$

$$= \sum_{\nu \in K_x} \Pr\left(\underset{j=1}{\overset{n}{\&}} M_{\nu j} \subseteq S(A_j) \subseteq X \dashv N_{\nu j}\right) , \quad (25)$$

and thus in Eq. (19), defining $\star$, $\underset{\nu \in K_x}{\text{Or}}$ is replaced by $\underset{\nu \in K_x}{\text{Or}}$ (disjoint), iff for all $x \in X$ and each pair $\nu_1, \nu_2 \in K_x$, with $\nu_1 \neq \nu_2$, there is an integer j, $1 \leq j \leq n$, such that either $M_{\nu_1, j} \cap N_{\nu_2, j} \neq \emptyset$ or $M_{\nu_2, j} \cap N_{\nu_1, j} \neq \emptyset$, such that Eq. (25) holds.

(ii) If also S is chosen so that $S(A_1), \dots, S(A_n)$ are all mutually statistically independent, for all $A_1, \dots, A_n \in F(X)$, then the characterizing equations Eq. (21) for case (1) and Eq. (23) for case 2 – in (i) become here, respectively

$$\phi_{\circledast}(A_1, \dots, A_n)(x) = \sum_{\nu \in K_x} \prod_{j=1}^{n} \omega'(\{\nu\}, A_j) , \quad (21')$$

$$\phi_{\circledast}(A_1, \dots, A_n)(x) = \sum_{\nu \in K_x} \prod_{j=1}^{n} \omega''(\{\nu\}, A_j) , \quad (23')$$

where ω' is given in Eq. (22) and ω'' in Eq. (24), for $B = \{\nu\}$.

■

Note that for Corollary 1 and 2, the construction of induced homomorphic $\star$, given $\circledast$, remains formally the same as in the general case as given in Eq. (19).

Consider now the case of generalized composition.

Theorem 6

Let $n \geq 1$ be arbitrary fixed and let S be a given choice function with all $(S(A_1), \dots S(A_n))$ assumed to have a specified consistent distribution.

(i) When $\star$ is a generalized composition, $\star$ is continuous, since, for any $x \in X$,

$$\star^{-1}(\mathcal{O}(x,1)) = \bigcup_{(a_1,\dots,a_n) \in g_x^{-1}(1)} \ \underset{j=1}{\overset{n}{\times}} \mathcal{O}(h_j(x), a_j) . \quad (26)$$

(ii) n-ary fuzzy set operation $\circledast$ induces a weak n-homomorphism relative to some ordinary set operation $\star$ being a generalized composition iff $\circledast$ has the following form:

$$\phi_{\circledast}(A_1,\dots,A_n)(x) = \sum_{\substack{(a_1,\dots,a_n) \in \\ \{0,1\} \times \cdots \times \{0,1\}}} (g_x(a_1,\dots,a_n) \cdot \Pr(x \in (\bigcap_{\substack{1 \leq j \leq n,\\ \text{such that } a_j = 1}} h_j^{-1}(S(A_j)) \dashv \bigcup_{\substack{1 \leq j \leq n,\\ \text{such that } a_j = 0}} h_j^{-1}(S(A_j))))) , \quad (27)$$

where

$g_x : \{0,1\} \times \cdots \times \{0,1\} \to \{0,1\}$ (n factors) .

(iii) By varying the distributions of $(S(A_1), \dots S(A_n))$ and the function g_x, the coefficients $g_x(a_1, \dots, a_n)$ can be arbitrarily chosen $\in \{0,1\}$, $x \in X$. There are 2^{2^n} such g_x's, for each $x \in X$.

(iv) If, S is such that $S(A_1), \dots, S(A_n)$ are always statistically independent, then Eq. (27) reduces to

$$\phi_{\star}(A_1, \dots, A_n)(x) = \sum_{\substack{(a_1,\dots,a_n) \in \\ \{0,1\} \times \cdots \times \{0,1\}}} (g_x(a_1,\dots,a_n) \cdot \prod_{\substack{1 \leq j \leq n\\ \text{such that}\\ a_j = 1}} \phi_{A_j}(h_j(x)) \cdot \prod_{\substack{1 \leq j \leq n\\ \text{such that}\\ a_j = 0}} (1 - \phi_{A_j}(h_j(x)))) \quad (28)$$

■

Finally, we treat the general situation of an n-homomorphic induced ordinary set operation being only measurable with respect to suitably chosen σ-algebras relative to topologies $\mathscr{V}$ and $\mathscr{T}$.

Consider again the fixed compact Hausdorff topological spaces $(G(X), \mathscr{T})$ and $(G(X) \times \cdots \times G(X), \mathscr{V})$ (n factors), and choice function S, where for any $A_1, \ldots, A_n \in F(X)$, $(S(A_1), \ldots, S(A_n))$ has a specified joint distribution. Thus, each marginal random subset $S(A_j)$ of X corresponds to probability space $(P(X), \mathscr{A}_j, q_j)$, as well as to probability space $(G(X), \mathscr{B}_j, p_j)$. Recall from section 2 the requirement that the σ-algebra $\mathscr{A}_j$ must contain all $C_{\{x\}}$, $x \in X$. Thus, $\mathscr{B}_j$ must contain all $\mathscr{O}(x,1)$, $x \in X$, and hence all countable unions and intersections of $\mathscr{O}(x,1)$, $x \in X$ and $\mathscr{O}(y,o)$, $y \in X$. Recall also that each $\mathscr{O}(x,1)$ and $\mathscr{O}(y,o)$ are closed-open-compact subsets of $G(X)$ with respect to $\mathscr{T}$; it follows that each $\mathscr{B}_j \supseteq \mathscr{B}_o$, the <u>Baire σ-algebra</u> generated by all compact - G_δ (i.e., compact and represented by an at most countable intersection of open) sets in $\mathscr{T}$. (See [8], sections 51, 52 for general background.) Note that any typical compact - G_δ set here is of the form $\mathscr{O}(Z,a)$, where $Z = (Z_j)_{j=1,\ldots,m} \in \overset{m}{\underset{j=1}{\times}} G(X)$ and $a = (a_j)_{j=1,\ldots,m} \in \overset{m}{\underset{j=1}{\times}} \{0,1\}$, where m may be a finite positive integer or $+\infty$. (See Halmos [8], Example 4, p. 219, for a related result.) If X is at most countably infinite, then since $\mathscr{T}$ will be separable, every compact set in $\mathscr{T}$ will also be a G_δ set and thus $\mathscr{B}$ the <u>Borel σ-algebra</u>, generated by all compact-open subsets of $\mathscr{T}$, will be the same as $\mathscr{B}_o$ ([8], Theorem E, p. 218). For X not countable, then generally $\mathscr{B}_o \subset \mathscr{B}$ (proper subclass). While $\mathscr{B}$ has certain desirable properties, it does not possess the regularity properties that $\mathscr{B}_o$ does. Analagous results hold for $(S(A_1), \ldots, S(A_n))$ and its corresponding probability space $(G(X) \times \cdots \times G(X), \mathscr{E}, p)$, where σ-algebras $\mathscr{E} \supseteq \mathscr{E}' \supseteq \mathscr{D}_o$, with, in general $\mathscr{D}_o \subset \mathscr{D}$, where $\mathscr{E}'$ is the σ-algebra generated by $\mathscr{B}_1 \times \cdots \times \mathscr{B}_n$; $\mathscr{D}_o$ is the Baire σ-algebra, relative to $\mathscr{V}$, generated by $\mathscr{B}_o \times \cdots \times \mathscr{B}_o$, and $\mathscr{D}$ is the Borel σ-algebra generated by $\mathscr{B} \times \cdots \times \mathscr{B}$. Thus, in order to utilize regularity properties, we will say a mapping ψ: $G(X) \times \cdots \times G(X) \to G(X)$ is (Baire) measurable iff $\psi^{-1}(\mathscr{B}_o) \subseteq \mathscr{D}_o$. (Clearly, if ψ is continuous, it is measurable.)

<u>Theorem</u> 7

Let $n \geq 1$ be arbitrarily fixed and S a given choice function with $(S(A_1), \ldots, S(A_n))$ having a consistent specified distribution for each $A_1, \ldots, A_n \in F(X)$, where, motivated by the previous discussion, it is assumed that the σ-algebra corresponding to each $S(A_j)$ is $\mathscr{B}_o$, and the σ-algebra corresponding to $(S(A_1), \ldots, S(A_n))$ is $\mathscr{D}_o$. Then n-ary fuzzy set operation $\circledast$ induces a weak n-homomorphism relative to some ordinary n-ary set operation $\star$ being measurable <u>iff</u> $\circledast$ has the following form, for all $x \in X$, $A_1, \ldots, A_n \in F(X)$:

$$\phi_{\circledast}(A_1, \ldots, A_n)(x) = \lim_{\ell \to +\infty} \phi_{\circledast_\ell}(A_1, \ldots, A_n)(x)$$
$$= \lim_{\ell \to +\infty} \phi_{\circledast'_\ell}(A_1, \ldots, A_n)(x), \quad (29)$$

where for each positive integer ℓ, $\circledast_\ell(A_1, \ldots, A_n)$ is defined from Eq. (18) with $\circledast$ replaced by $\circledast_\ell$ and each K_x by (finite) $K_{x,\ell}$, where the limit in Eq. (29) is uniform with respect to all $x \in X$, and where $\circledast'_\ell$ is defined from Eq. (18) with $\circledast$ replaced by $\circledast'_\ell$ and each K_x by $K'_{x,\ell} \overset{\mathrm{df}}{=} \bigcup_{t=\ell}^{+\infty} K_{x,t}$ (at most, countably infinite, and consequently the last part of Eq. (18) must be replaced by a limit). ■

6. SUMMARY AND CONCLUSIONS

Those n-ary fuzzy set operations have been characterized which induce weak homomorphic n-ary ordinary set operations between random sets, given: (1) any given choice function, i.e., mapping between $F(X)$ and $S(X)$ which picks for each given fuzzy subset of X an equivalent random subset of X up to one point coverages, and (2) any compatible stochastic process. In particular, most ordinary fuzzy set operations and their generalizations were shown to induce <u>naturally corresponding</u> operations among random sets. (For example, see Tables 1, 2, the remarks in Theorem 6, and Eq. (19).) When the choice function is either $S(A) = S_{U_A}(A)$ or $S(A) = T_A$, A an arbitrary fuzzy subset of X, a large class of operations exists (including essentially all of the usual ones and their generalizations) such that fuzzy sets formed from finite combination of such operations applied to more primitive fuzzy sets can be identified with corresponding random sets formed from similar combinations of ordinary set operations applied to their primitive random set components. Combining this result with a previous one ([16], as Theorems 6 and 7), allows for complete flexibility in using fuzzy and/or random sets as inputs in describing an unknown quantity V based on all of this information.

Future work will be directed towards applications to specific systems of fuzzy set operations and further investigation of the relations among choice functions, compatible stochastic processes over $P(X)$, and weak homomorphisms. Relative to the latter, it is of some interest to determine for a given n-ary ordinary set operation $\star$, what are the possible corresponding n-ary fuzzy set operations $\circledast$ which induce $\star$ homomorphically (weakly, for example), as the choice function and/or joint distributions of random sets are made to vary over all combinations. Conversely, for a fixed fuzzy set operation, what possible random set operations does it induce as the choice function and/or joint distribution of random sets vary?

[1] By a level set (or α - cut), we mean the ordinary set $\phi_A^{-1}[\alpha, 1] \subseteq X$, given fuzzy set membership function ϕ_A: $X \to [0,1]$, $0 \leq \alpha \leq 1$.

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